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1980 J. Phys. A: Math. Gen. 13 3779

(http://iopscience.iop.org/0305-4470/13/12/028)

J. Phys. A: Math. Gen. 13 (1980) 3779–3784. Printed in Great Britain

COMMENT

The renormalisation group in the large-n limit for the vectorial paramagnon problem at $T = 0$

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Received 2 June 1980

Abstract. We use the extension of the Ma RG approach to quantum functionals in the limit $n \to \infty$ to obtain non-perturbative information for quantum systems at $T = 0$, restricting ourselves to the n-vector paramagnon problem.

Recently, considerable attention has been devoted to the study of the critical behaviour of quantum systems at zero temperature in terms of an appropriate extension of the Wilson renormalisation group (RG) approach (Béal-Monod 1974a, b, Hertz 1976, Gerber and Beck 1977, De Cesare 1978, Busiello and De Cesare 1979, 1980). In many cases, for $T \to 0$ a dimensional crossover $d \to d + z$ appears in the sense that the critical exponents of a d-dimensional quantum system at $T = 0$ are identical to the one for a $(d + z)$-dimensional classical system, where the parameter z depends on the way in which the Matsubara frequencies enter the representative quantum functionals.

An unusual behaviour appears for bosonised systems at $T = 0$ (De Cesare 1978); it cannot be explained by means of a simple dimensional crossover, rather in terms of a more complex crossover process $(d, n) \to (d_c = d + z,\ n_c = n - 2z)$ involving the dimensionality of the order parameter too (Busiello and De Cesare 1979, 1980). However the previous results are valid only to the first or second order in $\epsilon = 4 - (d + z)$. Of course non-perturbative RG solutions should be very useful for clarifying the dimensional crossover phenomenon and physical interpretation of the unusual behaviour of bosonised systems at $T = 0$. For classical systems an exact non-perturbative realisation of the RG approach exists; this is the case when the number n of the order parameter components goes to infinity (Ma 1973). Therefore non-perturbative information for quantum systems at $T = 0$ can be obtained by using an appropriate extension of the Ma RG approach to quantum functionals in the limit $n \to \infty$. In this paper we restrict the extension to the n-vector paramagnon problem at $T = 0$ in the large-n limit. The results can be considered as a complement of the Hertz results to first order in $\epsilon = 1 - d$ (Hertz 1976), and can have some relevance also for the more realistic case of finite n (Hertz and Klenin 1977). The general version of the Ma RG approach for other quantum functionals and the analysis of the critical behaviour for several n-vector quantum systems at $T = 0$ for $n \to \infty$ will be given in full detail elsewhere.

For the description of interacting paramagnons in a system of itinerant electrons close to the ferromagnetic instability, a quantum functional $\mathscr{H}\{\psi\}$ can be derived (Hertz 1976, Hertz and Klenin 1977) by using the Hubbard–Stratonovich transformation

(Hubbard 1959, Stratonovich 1957). The partition function $\mathscr{Z}$ of the system is expressed as a functional integral of $\exp(-\mathscr{H}\{\psi\})$ over an n-vector variable $\psi(\boldsymbol{x},\tau)\equiv\{\psi^\alpha(\boldsymbol{x},\tau);\alpha=1,\ldots,n\}$ at each site $\boldsymbol{x}$. In terms of dimensionless quantities only,

$$\mathscr{Z}\propto\int\prod_{\substack{\alpha,\boldsymbol{q},\omega_n\\0<|\boldsymbol{q}|<1\\0<|\omega_n|<1}}\delta\psi^\alpha_{\boldsymbol{q}\omega_n}\,\mathrm{e}^{-\mathscr{H}\{\psi\}}\tag{1}$$

with

$$\mathscr{H}\{\psi\}=\frac{1}{2}\sum_{\substack{\boldsymbol{q},\omega_n\\0<|\boldsymbol{q}|<1\\0<|\omega_n|<1}}\left(q^2+\frac{|\omega_n|}{q}\right)|\psi_{\boldsymbol{q}\omega_n}|^2+\frac{1}{2}\sum_{\boldsymbol{x}}\int_0^\beta\mathrm{d}\tau\,U[\psi^2(\boldsymbol{x},\tau)]\tag{2}$$

where $\omega_n=2\pi n/\beta$ $(n=0,\pm1,\pm2,\ldots)$, $\beta=1/T$, $U(\psi^2)$ is a power series in ψ^2 and

$$\psi^\alpha(\boldsymbol{x},\tau)=\frac{1}{(\beta N)^{1/2}}\sum_{\substack{\boldsymbol{q},\omega_n\\0<|\boldsymbol{q}|<1\\0<|\omega_n|<1}}\psi^\alpha_{\boldsymbol{q}\omega_n}\,\mathrm{e}^{\mathrm{i}(\boldsymbol{q}.\boldsymbol{x}-\omega_n\tau)}\qquad(\alpha=1,\ldots,n),$$

$$\psi^2(\boldsymbol{x},\tau)=\sum_{\alpha=1}^{n}(\psi^\alpha(\boldsymbol{x},\tau))^2,\tag{3}$$

$$|\psi_{\boldsymbol{q}\omega_n}|^2=\sum_{\alpha=1}^{n}|\psi^\alpha_{\boldsymbol{q}\omega_n}|^2.$$

Note that both in wavevector and frequency spaces a natural cut-off is assumed (Hertz 1976).

As usual, the RG transformation R_b is globally defined by

$$\exp(-\mathscr{H}'\{\psi'\})\propto\left(\int\prod_{\alpha=1}^{n}\prod_{\boldsymbol{q}\omega_n}{}'\,\delta\psi^\alpha_{1\boldsymbol{q}\omega_n}\exp(-\mathscr{H}\{\psi_0+\psi_1\})\right)_{\psi_0^\alpha(\boldsymbol{x},\tau)\to\theta_b^{-1}\psi'^\alpha(b^{-1}\boldsymbol{x},b^{-z}\tau)}\tag{4}$$

where, as a first step, we have separated the paramagnon fields in two parts involving small wavevector components *and* frequencies and large wavevector components *or* frequencies, respectively:

$$\psi^\alpha(\boldsymbol{x},\tau)=\psi_0^\alpha(\boldsymbol{x},\tau)+\psi_1^\alpha(\boldsymbol{x},\tau),$$

$$\psi_0^\alpha(\boldsymbol{x},\tau)=\frac{1}{(\beta N)^{1/2}}\sum_{\substack{\boldsymbol{q}\omega_n\\0<|\boldsymbol{q}|<b^{-1}\\0<|\omega_n|<b^{-z}}}\psi^\alpha_{0\boldsymbol{q}\omega_n}\,\mathrm{e}^{\mathrm{i}(\boldsymbol{q}.\boldsymbol{x}-\omega_n\tau)},\tag{5}$$

$$\psi_1^\alpha(\boldsymbol{x},\tau)=\frac{1}{(\beta N)^{1/2}}\sum_{\boldsymbol{q}\omega_n}{}'\,\psi^\alpha_{1\boldsymbol{q}\omega_n}\,\mathrm{e}^{\mathrm{i}(\boldsymbol{q}.\boldsymbol{x}-\omega_n\tau)}.$$

In (4) and (5)

$$\sum_{\boldsymbol{q}\omega_n}{}'\ldots\equiv\sum_{\substack{0<|\boldsymbol{q}|<1\\b^{-z}<|\omega_n|<1}}\ldots+\sum_{\substack{b^{-1}<|\boldsymbol{q}|<1\\0<|\omega_n|<b^{-z}}}\ldots,$$

$$\prod_{\boldsymbol{q}\omega_n}{}'\ldots\equiv\prod_{\substack{0<|\boldsymbol{q}|<1\\b^{-z}<|\omega_n|<1}}\ldots\prod_{\substack{b^{-1}<|\boldsymbol{q}|<1\\0<|\omega_n|<b^{-z}}}\ldots\tag{6}$$

and

$$\theta_b=b^{-[2-\eta-(d+z)]/2},\qquad b\geqslant1.$$

Now we proceed in the same spirit as Ma (1973) for classical systems, and in the large-n limit we approximate

$$\psi^2(\boldsymbol{x}, \tau) \approx \psi_0^2(\boldsymbol{x}, \tau) + \rho \tag{7}$$

where

$$\rho = \frac{1}{\beta N} \sum_{q\omega_n}{}' N_{q\omega_n}, \qquad N_{q\omega_n} = |\psi_{1q\omega_n}|^2. \tag{8}$$

So, the multiple integral in (4) can be evaluated by transforming to the new variables $\{N_{q\omega_n}\}$. In the large-n limit one has

$$\int \prod_{\alpha=1}^{n} \prod_{q\omega_n}{}' \delta\psi_{1q\omega_n}^{\alpha} \exp(-\mathscr{H}\{\psi_0+\psi_1\}) \propto \exp\left[-\frac{1}{2}\sum_{q\omega_n}\left(q^2+\frac{|\omega_n|}{q}\right)|\psi_{0q\omega_n}|^2\right] \times \prod_{q\omega_n}{}' \int_0^{\infty} \mathrm{d}N_{q\omega_n} \exp\left(-\frac{1}{2}\sum_{x}\int_0^{\beta}\mathrm{d}\tau\, W(\psi_0^2, N_{q\omega_n})\right) \tag{9}$$

where

$$W(\psi_0^2, N_{q\omega_n}) = \frac{1}{\beta N}\sum_{q\omega_n}{}'\left[-n \ln N_{q\omega_n} + \left(q^2+\frac{|\omega_n|}{q}\right)N_{q\omega_n}\right] + U(\rho+\psi_0^2). \tag{10}$$

The last integral can be approximated by the maximum of its integrand, and we have

$$\int \prod_{\alpha=1}^{n} \prod_{q\omega_n}{}' \delta\psi_{1q\omega_n}^{\alpha} \exp(-\mathscr{H}\{\psi_0+\psi_1\}) \propto \exp\left\{-\left[\frac{1}{2}\sum_{q\omega_n}\left(q^2+\frac{|\omega_n|}{q}\right)|\psi_{0q\omega_n}|^2 + \frac{1}{2}\sum_{x}\int_0^{\beta}\mathrm{d}\tau\, \bar{W}\right]\right\} \tag{11}$$

where

$$\bar{W} = W(\psi_0^2, \bar{N}_{q\omega_n}) \tag{12}$$

and $\bar{N}_{q\omega_n}$, the solution of the equation $\partial W/\partial N_{q\omega_n} = 0$, is determined by the equations

$$\bar{N}_{q\omega_n} = \frac{n}{q^2+|\omega_n|/q+t(\bar{\rho}+\psi_0^2)}, \qquad \bar{\rho} = \frac{1}{\beta N}\sum_{q\omega_n}{}' \frac{n}{q^2+|\omega_n|/q+t(\bar{\rho}+\psi_0^2)}, \tag{13}$$

with $t(\psi^2) = \mathrm{d}U(\psi^2)/\mathrm{d}\psi^2$.

Finally, according to the transformation (4), we have

$$\mathscr{H} \xrightarrow{R_b} \mathscr{H}' = \frac{1}{2}\sum_{\substack{q'\omega'_n \\ 0<|q'|<1 \\ 0<|\omega'_n|<1}} \left(q'^2+\frac{|\omega'_n|}{q'}\right)|\psi'_{q'\omega'_n}|^2 + \frac{1}{2}\sum_{x'}\int_0^{\beta'}\mathrm{d}\tau'\, U'[\psi'^2(\boldsymbol{x}', \tau')] \tag{14}$$

where

$$q' = bq, \qquad \omega'_n = b^z\omega_n, \qquad x' = b^{-1}x, \qquad \tau' = b^{-z}\tau,$$
$$\beta' = b^{-z}\beta, \qquad \psi'(\boldsymbol{x}', \tau') = \theta_b\psi(\boldsymbol{x}, \tau), \qquad \psi'_{q'\omega'_n} = b^{-(2-\eta)/2}\psi_{q\omega_n}$$

and

$$U'(\psi'^2) = b^{(d+z)} W(b^{[2-\eta-(d+z)]}\psi'^2, \bar{N}_{q\omega_n}) \tag{15}$$

with $\eta = 0$ and $z = 3$.

The exact RG recursion relation, conveniently expressed in terms of $t(\psi^2)$, is (dropping the prime in ψ')

$$t'(\psi^2)=b^2t(\bar{\rho}+b^{-(d+1)}\psi^2), \qquad \bar{\rho}=\frac{n}{\beta N}\sum_{q\omega_n}{}'\left(q^2+\frac{|\omega_n|}{q}+\frac{t'}{b^2}\right)^{-1}. \tag{16}$$

Note that, apart from a different definition of $\bar{\rho}$, the RG equation $(16)_1$ is obtainable from the classical one with the dimensional shift $d\to d+3$. For temperature $T\neq 0$, where only the frequency $\omega_0=0$ contributes ($\beta'=b^{-z}\beta\to 0$ for $b\to\infty$ and $\Delta\omega'=2\pi/\beta'\to\infty$), it is easy to show that the Ma classical equations immediately follow from (16).

Let us consider now the case $T=0$ when all the frequencies contribute (quantum limit).

For $\bar{\rho}$ in (16) we have

$$\bar{\rho}=\frac{nK_d}{\pi}\left(\int_0^1 \mathrm{d}q\, q^d\int_{b^{-z}}^1\frac{\mathrm{d}\omega}{q^3+qt'/b^2+\omega}+\int_{b^{-1}}^1\mathrm{d}q\,q^d\int_0^{b^{-z}}\frac{\mathrm{d}\omega}{q^3+qt'/b^2+\omega}\right) \tag{17}$$

where $K_d=2^{1-d}\pi^{-d/2}/\Gamma(d/2)$.

The necessary condition for criticality, which determines the critical surface on which $t'(\psi^2)$ will approach a finite fixed point $t^*(\psi^2)$ as $b\to\infty$, is

$$t_1=t(N_c)=0 \tag{18}$$

where

$$N_c=\bar{\rho}(b=\infty)=\frac{nK_d}{\pi\,(d+1)}\left[\ln 2+\mathscr{B}\left(\frac{d+1}{3}\right)\right], \qquad \mathscr{B}(\mu)=\int_0^1\mathrm{d}x\frac{x^{\mu-1}}{x+1}, \qquad \operatorname{Re}\mu>0. \tag{19}$$

It is to be noticed that in the present quantum case, $N_c(d)>0$ is defined for any $d>0$, in contrast with the classical value $N_c=nK_d/(d-2)$.

For t near or on the critical surface, for $d>0$ and $b\gg 1$, from equation (17) we obtain the relation

$$\frac{\psi^2}{N_c}=1+\phi b^{d-1}-\frac{d+1}{\ln 2+\mathscr{B}[(d+1)/3]}\left(\int_0^1\mathrm{d}q\,q^d\ln\frac{q^3+1}{q^3+qt'+1}+\int_1^b\mathrm{d}q\,q^d\ln\frac{q^2}{q^2+t'}\right) \tag{20}$$

where the parameter ϕ is defined so that

$$(N-N_c)/N_c=\phi/b^2\ll 1, \qquad N=\bar{\rho}+b^{-(d+1)}\psi^2. \tag{21}$$

Equation (20), which differs from the corresponding classical one (Ma 1973), can be utilised to determine the finite fixed points of the transformation R_b, i.e. the solutions of the integral equation $t^*(\psi^2)=b^2t^*(\bar{\rho}^*+b^{-(d+1)}\psi^2)$ with $\bar{\rho}^*=\bar{\rho}(t^*)$, and the critical behaviour of the n-vector paramagnon problem in the large-n limit.

For $d<1$ there exists a non-trivial fixed point $t^*(\psi^2)<\infty$ determined as a function of ψ^2 by the equation

$$\frac{\psi^2}{N_c}=1-\frac{d+1}{\ln 2+\mathscr{B}[(d+1)/3]}\left(\int_0^1\mathrm{d}q\,q^d\ln\frac{q^3+1}{q^3+qt^*+1}+\int_1^\infty\mathrm{d}q\,q^d\ln\frac{q^2}{q^2+t^*}\right). \tag{22}$$

Of course the dimensionality region $0<d<1$ has no physical relevance (Hertz 1976). However, it is also interesting to determine the critical exponents for $d<1$ in order to verify explicitly that in the Ma quantum RG approach also the crossover $d \to d+3$ is realised.

If $t_1 = t(N_c) \neq 0$, for $d<1$ and $b \gg 1$, equations (20) and (22) give

$$-\frac{d+1}{\ln 2+\mathscr{B}[(d+1)/3]}\left(\int_0^1 \mathrm{d}q\, q^d \ln\frac{q^3+qt^*+1}{q^3+qt'+1}+\int_1^\infty \mathrm{d}q\, q^d \ln\frac{q^2+t^*}{q^2+t'}\right)$$
$$\approx b^{d+1}(t_1/U_c)[1+\mathrm{O}(t_1)]+\mathrm{O}(b^{d-1}), \qquad U_c = N_c(\mathrm{d}t/\mathrm{d}N)_{N=N_c}, \tag{23}$$

and for $t'-t^* \ll 1$ we have

$$t'-t^* \propto t_1 b^{\lambda_1}+\mathrm{O}(\lambda_2) \tag{24}$$

with

$$\lambda_1 = d+1, \qquad \lambda_2 = d-1<0. \tag{25}$$

Finally, the critical exponents are

$$\eta=0, \qquad z=3, \qquad \nu=\frac{1}{\lambda_1}=\frac{1}{d+1},$$
$$\gamma=\frac{2}{d+1}, \qquad \alpha=\frac{d-1}{d+1}, \qquad \beta=\frac{1}{2}, \qquad \delta=\frac{d+5}{d+1}, \tag{26}$$

which are just obtainable from the classical one (spherical model) for $2<d<4$, with the dimensionality shift $d\to d+3$.

Let us now consider the case $d>1$. Firstly, assuming $t_1=0$, simple considerations based on equation (20) indicate that we must have $t'b^{d-1}=\mathrm{O}(1)$ for $b\to\infty$, and therefore the Gaussian fixed point $t^*(\psi^2)=0$ is approached for $d>1$. Then, if $t_1\neq 0$ and $b\gg 1$, for small t' we have from equation (20)

$$\frac{\psi^2}{N_c}\simeq 1+\left(-\frac{b^2 t_1}{U_c}+\mathrm{O}(b^{-2})\right)b^{d-1}+t'\frac{(d+1)/(d-1)}{\ln 2+\mathscr{B}[(d+1)/3]}(b^{d-1}-1), \tag{27}$$

and solving for t', we obtain

$$t' \sim t_1 b^{\lambda_1}+\mathrm{O}(b^{\lambda_2}) \tag{28}$$

with

$$\lambda_1=2, \qquad \nu=1/\lambda_1=\tfrac{1}{2}, \qquad \lambda_2=1-d<0. \tag{29}$$

So, for $d>1$ the Landau classical exponents are correct.

Finally, in the marginal case $d=1$, as for classical systems, we find that on the critical surface

$$t' \xrightarrow{b\to\infty} t^* \propto \lim_{b\to\infty}(\ln b)^{-1}=0 \tag{30}$$

and also at $d=1$ there exists a trivial fixed point, but logarithmic corrections arise, as easily follows from equation (20). In conclusion, for the n-vector paramagnon problem, we find that in the large-n limit also the dimensional crossover $d\to d+3$ occurs for $T\to 0$ with the shift $d^*=4\to d^*=1$ of the borderline dimension d^*, in agreement with the perturbative RG analysis by Hertz (1976). Finally, a relevant consequence of the dimensional crossover is to extend, at any dimensionality $d>0$, the RG in the large-n limit for the paramagnon problem at $T=0$, in contrast with the Ma classical analysis which is valid for $d > 2$. This is a characteristic common to other quantum systems for which we have $d^* > 1$ ($d^* = 2, 3, \ldots$), and gives the possibility of exploring their RG properties in the large-n limit also for dimensionality $d \leqslant 2$.

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